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Nonlinear convection in high vertical channels

By C. NORMAND
CEN-Saclay, 91191 Gif-sur-Yvette Cedex, France

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Application of Landau's ideas to the theory of weakly nonlinear instabilities shows that the amplitude of the unstable modes behaves as the square root of the reduced control parameter ϵ, its critical value being $\epsilon = 0$. When applied to cellular structures the theory has been improved by taking into account the slow spatial variations of the amplitude and phase of the unstable modes. Until now the case of thermoconvective instabilities in high vertical channels has not been studied using this approach. In high vertical structures the nonlinear terms disappear in the limit of an infinite height, and the supercritical behaviour requires a specific treatment. It differs from the standard analysis valid for the case of fluid layers of infinite horizontal extent, where the nonlinearities and the finite-size effects are disconnected. In the limit of high aspect ratios (height $\gg$ horizontal extent) we have derived an amplitude equation for convective systems where the nonlinear terms contain derivatives at the lowest order. As a consequence the amplitude equation cannot be put into a variational form and the stability of the stationary solutions cannot be deduced from an ordering in decreasing values of a Lyapunov functional.

1. Introduction

In laboratory experiments, convection always takes place in closed cavities, but most of the theoretical studies deal with the assumption of an infinite extent in at least one direction. This simplification leads to two typical situations: convective layers of infinite horizontal extent or vertical channels of infinite height. In both cases the temperature gradient is parallel to gravity. The fundamental difference between these two cases is apparent when investigating the nonlinear regime. In infinitely long vertical channels we are faced to the problem of how to generate nonlinear terms in the Boussinesq equations. For such systems the assumption of an infinite height is equivalent to a linearization of the governing equations where the two relevant physical quantities remain the vertical component of the convective velocity v_z and the temperature θ, which are both independent of the vertical coordinate. In the following we shall refer to this approximation as the Ostroumov limit (Ostroumov 1947). Then the convective currents consist of alternatively upward and downward flow so that the mass-conservation condition is satisfied. This is quite different to what happens in infinitely large horizontal convective layers, where the perturbative expansion method (Sorokin 1954; Gorkov 1957; Malkus & Veronis 1958; Schlüter, Lortz & Busse 1967) has been, among others, a powerful tool for the treatment of the nonlinearities.

In long vertical channels we have studied the nonlinearities due to the presence of horizontal boundaries limiting the fluid at the top and the bottom. In §§2 and 3 two cases will be examined: convection between vertical plates of large but finite height and the convection in long vertical cylinders with special emphasis given to

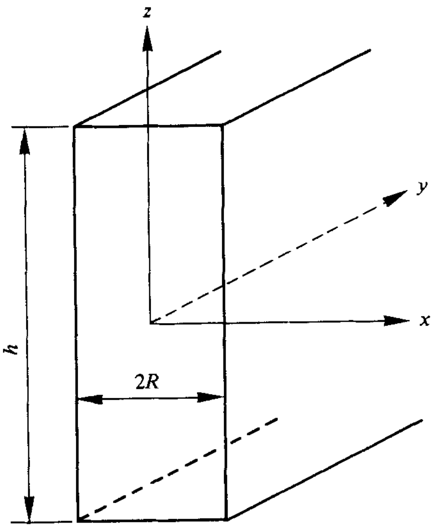


FIGURE 1. Plane vertical layer. Coordinate axes.

the case of circular cylinders. For closed cavities it is exceptional to get analytic solutions of the linearized equations, and the solution depends on the boundary conditions on the lateral and horizontal walls. For instance, in finite right-circular cylinders with free horizontal boundary, Catton & Edwards (1970) derived exact analytical results, whereas for rigid horizontal boundary only approximate solutions are known. When the nonlinear terms are included and whatever the boundary conditions are, the equations can only be solved by approximate methods (Charlson & Sani 1975; Liang, Vidal & Acrivos 1969). We shall describe in the following an alternate analytic approach valid in the weakly nonlinear regime by making use of an expansion method having some connection with the theory developed for convective flows by Segel (1969) and Newell & Whitehead (1969). Before going further we must introduce an important parameter λ, the aspect ratio of the cell, which is the ratio of the height of the cell to its characteristic horizontal length (the radius for a cylinder or the gap between parallel vertical plates). When λ is large the system is divided into an inner region, where the solutions are in first approximation those for a system of infinite vertical extent, and two boundary layers at the top and the bottom, where the horizontal components of the fluid velocity cannot be neglected. The solutions in the boundary layers can be considered as corrections to the Ostroumov solution, and will be accounted for by a perturbative expansion in powers of λ^{-1}. The solution for the whole system results in a matching between the two regions. The calculation will be given in detail for two systems: the plane vertical layer (§2) and the vertical cylinder (§3).

2. The plane vertical layer

We consider a vertical gap of width $2R$ confined between two plates of height h and infinite along the horizontal direction y (figure 1). The temperatures at the top and bottom boundary are held constant and equal to T_1 and T_2 with $T_1 < T_2$. With the origin of the Cartesian coordinates $Oxyz$ located in the midheight horizontal plane, the temperature of the fluid is

$$\tfrac{1}{2}(T_1+T_2)+(T_1-T_2)\frac{z}{\lambda}+\frac{\theta\chi\nu}{\alpha g R^3},$$

where χ, ν, α and g denote respectively the thermal diffusivity, the kinematic viscosity, the thermal expansion coefficient and the acceleration due to gravity. Let R be the unit of distance and R^2/χ the unit of time; the velocity is $(\chi/R)\,\boldsymbol{v}$. With these definitions the governing equations reduce to the Oberbeck–Boussinesq equations:

$$\nabla\cdot\boldsymbol{v} = 0, \tag{2.1a}$$

$$Pr^{-1}\left(\frac{\partial\boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\right) = -\nabla p+\Delta\boldsymbol{v}+\theta\boldsymbol{e}, \tag{2.1b}$$

$$\frac{\partial\theta}{\partial t}+(\boldsymbol{v}\cdot\nabla)\,\theta = \Delta\theta+Ra\,v_z, \tag{2.1c}$$

where $\boldsymbol{e} = (0,0,1), v_z = \boldsymbol{e}\cdot\boldsymbol{v}$, and p is the reduced pressure. The Rayleigh number and the Prandtl number are

$$Ra = \frac{\alpha g\Delta TR^4}{\nu\chi h}, \quad Pr = \frac{\nu}{\chi}.$$

We take the upper and lower boundaries to be rigid and conducting, so that

$$\boldsymbol{v} = \theta = 0 \quad \text{on} \quad z = \pm\lambda.$$

The boundary conditions on the vertical plates will be given later. The flow of a viscous liquid between vertical parallel plates may be described by approximate equations of motion which have a simple form. Two kinds of disturbances have been discussed in the past.

(i) Plane motions in which the velocity is vertical and all quantities are independent of the coordinate y have been examined by Ostrach (1955) and Yih (1959).

(ii) The assumption that the component of velocity normal to the vertical plates vanishes has been used by Wooding (1960). This model gives a good picture of the convective behaviour in a Hele-Shaw cell.

It can be shown that the most unstable disturbances are of the form (ii). Nevertheless the case (i) bears an analogy with the case of a vertical cylinder discussed in §3 and has the advantage of allowing entirely analytical calculations. Thus to illustrate our method we have chosen to treat this case.

The incompressibility condition (2.1a) is automatically satisfied by introducing the stream function ϕ:

$$v_x = \frac{\partial\phi}{\partial z}, \quad v_z = -\frac{\partial\phi}{\partial x}. \tag{2.2}$$

After elimination of the pressure in (2.1), one gets

$$Pr^{-1}\left(\frac{\partial\Delta v_z}{\partial t}+N_v\right) = \Delta^2 v_z+\frac{\partial^2\theta}{\partial x^2}, \tag{2.3a}$$

with

$$N_v = -\partial_x(\boldsymbol{v}\cdot\nabla)\,\Delta\phi. \tag{2.3b}$$

Equation (2.3a) must be solved together with (2.1c). To recover the classical results of Ostroumov in the limit, $h\rightarrow\infty$, it is appropriate to make the change of variable $z = \lambda Z$ and to expand the quantities ϕ, θ and Ra so that

$$\phi = \phi^{(0)}+\tilde{\phi}, \tag{2.4a}$$

$$\theta = \theta^{(0)}+\tilde{\theta}, \tag{2.4b}$$

$$Ra = Ra^{(0)}+\widetilde{Ra}, \tag{2.4c}$$

where the superscript 0 refers to quantities relative to the infinite problem, whereas the tilde denotes that the corresponding quantity admits an expansion in power of λ^{-1}. Similar expansions hold for v_z and v_x. After substituting the set of expressions (2.4) in (2.3a) and (2.1c), we collect the terms according to their order in λ^{-1}. The lowest order gives

$$(\partial_x^4 - Ra^{(0)})\, v_z^{(0)} = 0, \tag{2.5a}$$

with

$$\theta^{(0)} = -\partial_x^2\, v_z^{(0)}. \tag{2.5b}$$

This is precisely the equation for an infinite plane vertical layer. The solutions for the odd modes are as follows:

(a) Conducting lateral boundary:

$$v_z^{(0)} = A \sin\gamma x, \quad \theta^{(0)} = A\gamma^2 \sin\gamma x, \tag{2.6}$$

with for the lowest mode

$$Ra^{(0)} = \gamma^4 = \pi^4,$$

and A is a constant.

(b) Insulating lateral boundary:

$$v_z^{(0)} = A\left(\frac{\sin\gamma x}{\sin\gamma} - \frac{\operatorname{sh}\gamma x}{\operatorname{sh}\gamma}\right), \quad \theta^{(0)} = \gamma^2 A\left(\frac{\sin\gamma x}{\sin\gamma} + \frac{\operatorname{sh}\gamma x}{\operatorname{sh}\gamma}\right), \tag{2.7}$$

with $\operatorname{th}\gamma = -\operatorname{tg}\gamma$, and $\gamma = 2.365$ for the lowest mode. The even modes require a special discussion because the flux-closure condition imposes an additional constraint on the velocity:

$$\int_{-1}^{+1} v_z(x)\,\mathrm{d}x = 0. \tag{2.8}$$

Integrating (2.1c) with respect to x from -1 to 1, which reduces it to

$$\frac{\partial^2\theta^{(0)}}{\partial x^2} + Ra\, v_z^{(0)} = 0, \tag{2.9}$$

and making use of (2.8), we obtain

$$\frac{\partial\theta}{\partial x} = 0 \quad \text{at} \quad x = \pm 1.$$

Therefore the solutions of (2.5) are in this case the same for either type of thermal boundary condition:

$$v_z^{(0)} = A\left(\frac{\cos\gamma x}{\cos\gamma} - \frac{\operatorname{ch}\gamma x}{\operatorname{ch}\gamma}\right), \quad \theta^{(0)} = A\gamma^2\left(\frac{\cos\gamma x}{\cos\gamma} + \frac{\operatorname{ch}\gamma x}{\operatorname{ch}\gamma}\right), \tag{2.10}$$

with $\operatorname{th}\gamma = \operatorname{tg}\gamma$, and $\gamma = 3.927$ for the lowest mode. The presence of boundaries at the top and bottom of the vertical layers makes a significant difference to this statement. However, provided that $\lambda \to \infty$, the form $v_z^{(0)} = Aw(x)$, $\theta^{(0)} = A\Theta(x)$, $\phi^{(0)} = A\Phi(x)$ may be adapted to the finite problem by allowing A to be a slowly varying function of Z as well as time. Then to the second order of perturbation one gets

$$[\partial_x^4 - Ra^{(0)}]\,\tilde{v}_z = (1 + Pr^{-1})\,A_t\,\partial_x^2 w + Pr^{-1}\,N_v - N_\theta + \widetilde{Ra}\,Aw - 3\lambda^{-2}A_{ZZ}\,\partial_x^2 w, \tag{2.11}$$

where

$$N_\theta = \frac{\partial\phi^{(0)}}{\partial z}\frac{\partial\theta^{(0)}}{\partial x} - \frac{\partial\phi^{(0)}}{\partial x}\frac{\partial\theta^{(0)}}{\partial z} = \lambda^{-1}AA_Z[\Phi\,\partial_x\Theta - \Theta\,\partial_x\Phi],$$

$$N_v = \lambda^{-1}N_v^{(1)} + \lambda^{-3}N_v^{(3)},$$

with $$N_v^{(1)} = -\partial_x(\phi_Z \partial_x - \phi_x \partial_Z)\,\partial_x^2 \phi, \quad N_v^{(3)} = -\partial_x(\phi_Z \partial_x - \phi_x \partial_Z)\,\partial_Z^2 \phi.$$

In the foregoing, except for the components of the velocity, v_x, v_y and v_z, the subscript denotes derivative with respect to the corresponding variable (i.e. $A_Z = \partial A/\partial Z$). Equation (2.11) is solved provided that its right-hand side is orthogonal to the kernel of the homogeneous adjoint operator. Neglecting all the terms of order higher than λ^{-2} in (2.11), the compatibility condition is expressed as

$$(1+Pr^{-1})\, I_1 A_t = 3\lambda^{-2} A_{ZZ} I_1 + A\, \widetilde{Ra}\, I_2 + \lambda^{-1} A A_Z (I_3 + Pr^{-1} I_4), \qquad (2.12)$$

where

$$I_1 = \int_{-1}^{+1} |\Theta(x)|^2\,\mathrm{d}x, \quad I_2 = \int_{-1}^{+1} w(x)\,\Theta(x)\,\mathrm{d}x, \qquad (2.13a,b)$$

$$I_3 = -\int_{-1}^{+1} \Theta(x)[\Phi(x)\,\partial_x\Theta - \Theta(x)\,\partial_x \Phi]\,\mathrm{d}x, \quad I_4 = -\int_{-1}^{+1} \Theta(x)[\Phi\,\partial_x^4\Phi - \Phi_{xx}^2]\,\mathrm{d}x. \qquad (2.13c,d)$$

The two quantities I_1 and I_2 are positive whereas I_3 and I_4 vanish when the odd modes are considered and take a finite value for the even modes:

$$I_3 = 2I_4 = \tfrac{32}{5}\gamma^3\,\mathrm{th}^3\gamma. \qquad (2.14)$$

In this latter case, $I_3 \neq 0$, $I_4 \neq 0$, all the terms in equation (2.12) can be made of the same order of magnitude in λ by an appropriate normalization. Let

$$\widetilde{Ra} = 3I_1 I_2^{-1} \lambda^{-2}\delta, \quad t = \tfrac{1}{3}\lambda^2(1+Pr^{-1})\,\tau, \qquad (2.15a,b)$$

$$A \to 3I_1(I_3 + Pr^{-1} I_4)^{-1}\,\lambda^{-1} A. \qquad (2.15c)$$

Then instead of (2.12) we get

$$A_{ZZ} + \delta A + A A_Z = A_\tau \quad (-1 < Z < +1) \qquad (2.16)$$

or

$$A_{\xi\xi} + A + A A_\xi = A_\tau \quad (-\delta^{\frac{1}{2}} < \xi < +\delta^{\frac{1}{2}}), \qquad (2.17)$$

where $\xi = \delta^{\frac{1}{2}} Z$, A is scaled with $\delta^{\frac{1}{2}}$ and τ with δ^{-1}. The derivation of the boundary conditions associated with this equation is given in Appendix A. We get the result that $A = 0$ at the boundaries. This corresponds to the condition $v_z = 0$, which is more restraining than $v_x = 0$, since $v_x = O(\lambda^{-1})$. All the parameters have been eliminated in (2.17).

We now set $A_\tau = 0$, and write (2.17) in the form of a first-order system:

$$A_\xi = B, \quad B_\xi = -A(B+1). \qquad (2.18a,b)$$

Multiplying (2.18*a*) by A and (2.18*b*) by $B(B+1)^{-1}$, we obtain after addition and integration

$$\tfrac{1}{2}A^2 + B - \log(B+1) = C,$$

where C is the constant of integration. It follows from the McHarg theorem (1947) that the trajectories in the (A, B)-plane are closed curves provided that the boundary value of B is less than one. Closed circles are the trajectories of the linear problem for which

$$\delta = \pi^2 \times \begin{cases} \tfrac{1}{4}(2n+1)^2 & \text{(even solutions)}, \\ n^2 & \text{(odd solutions)}. \end{cases}$$

A numerical solution of (2.18) using a space discretization gives the form of A (figure 2). We see that $\delta^{\frac{1}{2}} > \frac{1}{2}\pi$ for a solution to be possible. When $\delta^{\frac{1}{2}}$ is close to $\frac{1}{2}\pi$,

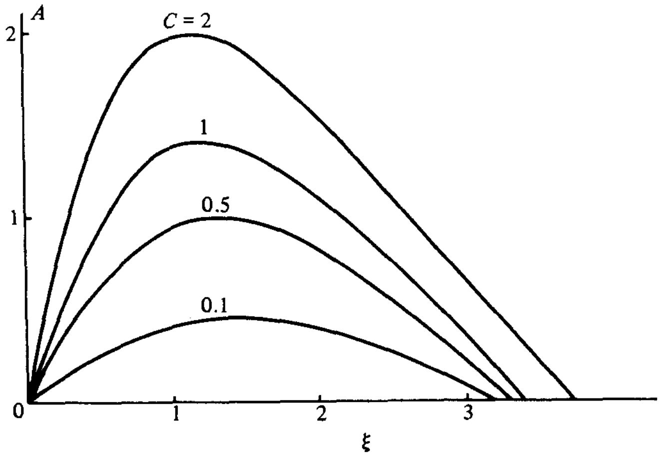


FIGURE 2. Numerical solutions of $A_{\xi\xi}+A+AA_{\xi} = 0$ with $A = 0$ at $\xi = \pm\delta^{\frac{1}{2}}$ (solutions are impossible for $\delta^{\frac{1}{2}} < \frac{1}{2}\pi$).

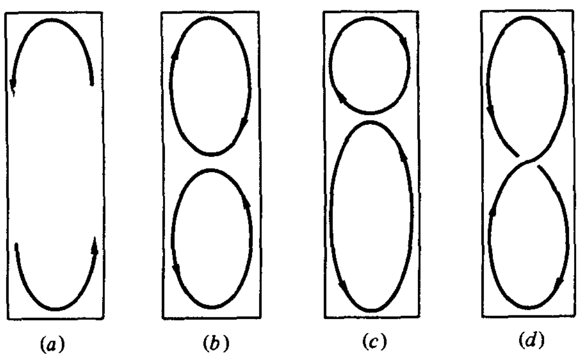


FIGURE 3. (*a*) and (*b*) schematic representation is the (x, z)-plane of the two lowest vertical modes having a constant phase. (*c*) Representation of a mode–mode coupling between (*a*) and (*b*). (*d*) Vertical mode with a non-constant phase. Such patterns in the form of a figure of eight have been observed in high-evaporation-rate cooling cells.

$A(\xi)$ is sinusoidal and of small amplitude, and as C increases, the maximum of A moves to the left. The non-symmetrical form of A suggests that the configuration of the convective flow results from a mixing between the odd and even eigenmodes of the linear problem (figure 3). If one tries to fit the numerical solution by a superposition of the two lowest modes,

$$A = A_1 \cos\tfrac{1}{2}\pi Z + A_2 \sin \pi Z, \tag{2.19}$$

then the Galerkin approximation gives

$$A_2 = -\frac{4}{\pi}\left(\delta - \frac{\pi^2}{4}\right), \quad A_1^2 = \frac{4}{\pi}(\delta-\pi^2)\,A_2. \tag{2.20}$$

It has been checked for $\delta^{\frac{1}{2}} = 1.7$ and 1.85 that the maximum value of A given by (2.20) is consistent with the corresponding numerical values deduced from figure 2.

We shall come back now to the case where $I_3 = I_4 = 0$ in (2.12), and then the nonlinear terms in AA_Z disappear from the differential equation satisfied by A. To introduce the nonlinearities correctly we shall proceed in a different way, and in particular we must drop the condition $A = O(\lambda^{-1})$ that we have introduced in (2.15*c*). The method consists in expanding $\tilde{v}_z$, $\tilde{\theta}$ and $\widetilde{Ra}$ in powers of λ^{-1}:

$$\tilde{v}_z = v_z^{(1)}\lambda^{-1} + v_z^{(2)}\lambda^{-2} + \dots, \tag{2.21a}$$

$$\tilde{\theta} = \theta^{(1)}\lambda^{-1} + \theta^{(2)}\lambda^{-2} + \dots, \tag{2.21b}$$

$$\widetilde{Ra} = Ra^{(1)}\lambda^{-1} + Ra^{(2)}\lambda^{-2} + \dots. \tag{2.21c}$$

After substitution of (2.21*a–c*) in (2.11) and equating terms of the same order in λ^{-1} we get to the first order in λ^{-1}

$$[\partial_x^4 - Ra^{(0)}]\, v_z^{(1)} = N^{(1)} + Ra^{(1)} v_z^{(0)}, \tag{2.22}$$

where $N^{(1)}$ is the first term in the λ^{-1} expansion of the nonlinear terms:

$$Pr^{-1} N_v - N_\theta = \lambda^{-1} N^{(1)} + \lambda^{-2} N^{(2)} + \dots.$$

The derivation of the evolution equation for the amplitude of the even modes is greatly simplified by considering the case of conducting lateral boundary for which $v_z^{(0)}$ and $\theta^{(0)}$ are given by (2.6). The stream function $\phi^{(0)}$ is determined from

$$v_z^{(0)} = -\partial_x \phi^{(0)}$$

and takes the value

$$\phi^{(0)} = \frac{A}{\gamma}(\cos\gamma x + 1), \tag{2.23}$$

which allows $v_x = 0$ to be satisfied at $x = \pm 1$.

The solvability condition for (2.22) is

$$Ra^{(1)} = \frac{\int N^{(1)} \Theta \, \mathrm{d}x}{\int v_z^{(0)} \Theta \, \mathrm{d}x}. \tag{2.24}$$

The numerator is proportional to I_3 and I_4, which are null: therefore we conclude that $Ra^{(1)} = 0$ and we seek the solution of (2.22) under the form

$$v_z^{(1)} = AA_Z w^{(1)}(x), \tag{2.25}$$

where $w^{(1)}$ satisfies

$$[\partial_x^4 - Ra^{(0)}]\, w^{(1)} = -\gamma^2[1 + (1 + Pr^{-1})\cos\gamma x], \tag{2.26}$$

whose solution is

$$w^{(1)} = \frac{1}{\gamma^2}(1 + \tfrac{1}{4}(1 + Pr^{-1})\,\gamma x \sin\gamma x + a\cos\gamma x + b\,\mathrm{ch}\,\gamma x), \tag{2.27}$$

where the homogeneous part of the solution depends on two constants a and b. From (2.2) we get

$$\phi^{(1)} = -\frac{AA_Z}{\gamma^2}\left[x + \frac{(1 + Pr^{-1})}{4\gamma}(-\gamma x\cos\gamma x + \sin\gamma x) + \frac{1}{\gamma}(a\sin\gamma x + b\,\mathrm{sh}\,\gamma x)\right], \tag{2.28}$$

and the boundary conditions

$$v_x = v_z = 0 \quad \text{at} \quad x = \pm 1$$

allow us to determine the constants a and b:

$$a = 1 - \frac{(5 + Pr^{-1})\gamma}{4\,\mathrm{th}\,\gamma}, \quad b = -\frac{(5 + Pr^{-1})\gamma}{4\,\mathrm{sh}\,\gamma}.$$

At the same order in λ^{-1} the temperature is given by

$$\theta^{(1)} = -AA_Z(-\tfrac{1}{4}(1+Pr^{-1})\gamma x \sin\gamma x + (\tfrac{1}{2}(1-Pr^{-1})-a)\cos\gamma x + b\,\mathrm{ch}\,\gamma x + c), \quad (2.29)$$

with

$$c = \frac{1}{2}\left[\frac{(5+Pr^{-1})\gamma}{\mathrm{th}\,\gamma} - (1+Pr^{-1})\right].$$

Collecting now all the terms which are of order λ^{-2} in (2.11), we get

$$[\partial_x^4 - Ra^{(0)}]\,v_z^{(2)} = (Pr^{-1}+1)\,\Theta^{(0)}A_t + N^{(2)} + Ra^{(2)}v_z^{(0)} - 3A_{ZZ}\,\partial_x^2 w^{(0)}. \quad (2.30)$$

and it is the solvability condition for this equation which gives the differential equation to be satisfied by the amplitude A. The nonlinear term in (2.30),

$$N^{(2)} = -Pr^{-1}\partial_x \sum_{\substack{i+j=1\\ i,j=0,1}} (\phi_Z^{(i)}\partial_x - \phi_x^{(i)}\partial_Z)\,\phi_{xx}^{(j)} - \sum_{\substack{i+j=1\\ i,j=0,1}} (\phi_Z^{(i)}\Theta_x^{(j)} - \phi_x^{(i)}\Theta_Z^{(j)}), \quad (2.31)$$

takes the form

$$N^{(2)} = A^2 A_{ZZ} N^{(2,1)} + AA_Z^2 N^{(2,2)}, \quad (2.32)$$

and consequently the equation for A is

$$A_{ZZ} + \delta A + \mu_1 A^2 A_{ZZ} + \mu_2 AA_Z^2 = A_\tau \quad (-1 < Z < +1), \quad (2.33)$$

where $\delta = Ra^{(2)}I_2(3I_1)^{-1}$ and

$$\mu_1 = (3I_1)^{-1}\int_{-1}^{+1} N^{(2,1)}\Theta^{(0)}\,\mathrm{d}x, \quad \mu_2 = (3I_1)^{-1}\int_{-1}^{+1} N^{(2,2)}\Theta^{(0)}\,\mathrm{d}x.$$

For a conducting lateral boundary the expressions for the μ_i are

$$\mu_1 = \tfrac{1}{2}\gamma^2[5u - 4 - \tfrac{1}{5}Pr^{-1}(16 - 15u + 2Pr^{-1})],$$
$$\mu_2 = \tfrac{1}{4}\gamma^2\{5u - \tfrac{23}{2} - Pr^{-1}[4u + \tfrac{11}{10} + Pr^{-1}(\tfrac{19}{10} + u)]\},$$

with $u = \gamma/\mathrm{th}\,\gamma$.

It must be noticed that in the limit $Pr \to \infty$ the coefficients μ_1 and μ_2 are always positive, and then (2.33) can be written

$$A_\tau = A_{\xi\xi} + A + A^2 A_{\xi\xi} + \mu AA_\xi^2 \quad (-\delta^{\frac{1}{2}} < \xi < \delta^{\frac{1}{2}}), \quad (2.34a)$$

where $\mu = \mu_2/\mu_1$, and A is scaled with $\mu_1^{-\frac{1}{2}}$.

In this limit $Pr \to \infty$ the coefficient μ takes its values in the range $0.1813 < \mu < 0.5$ when $\pi < \gamma < \infty$. In the opposite limit $Pr = 0$ the coefficients μ_1 and μ_2 are both negative, with the values

$$\mu_1 = -\tfrac{1}{5}\gamma^2, \quad \mu_2 = -\tfrac{1}{4}\gamma^2(\tfrac{19}{10} + u),$$

and (2.33) takes the form

$$A_\tau = A_{\xi\xi} + A - A^2 A_{\xi\xi} - \mu AA_\xi^2, \quad (2.34b)$$

with $\mu = |\mu_2|/|\mu_1|$. For $\gamma = \pi$ the numerical value of μ is 6.3. For intermediate values of the Prandtl number the coefficients μ_1 and μ_2 have opposite signs, leading to an evolution equation of the type

$$A_\tau = A_{\xi\xi} + A + A^2 A_{\xi\xi} - \mu AA_\xi^2. \quad (2.34c)$$

For example, the above equation is valid in the range $0.2 < Pr < 3.5$, when $\gamma = \pi$. Equations (2.34*a–c*) give rise to different convective behaviours, which are now examined successively.

We set $A_\tau = 0$ and write (2.34*a*) in the form of a first-order system:

$$\left.\begin{aligned} A_\xi &= B, \\ B_\xi &= -A\frac{1+\mu B^2}{1+A^2} \end{aligned}\right\} \quad (-\delta^{\frac{1}{2}} < \xi < \delta^{\frac{1}{2}}). \qquad \begin{aligned}(2.35a)\\(2.35b)\end{aligned}$$

Multiplying (2.35*a*) by $A/(1+A^2)$ and (2.35*b*) by $B/(1+\mu B^2)$, we obtain after addition and integration

$$\log(1+A^2)+\frac{1}{\mu}\log(1+\mu B^2) = Q, \tag{2.36}$$

where Q is the constant of integration. It must be noticed that for some peculiar values of μ, elliptic integrals are involved in the solution for $A(\xi)$. For $\mu = 1$, solutions exist provided that $\delta^{\frac{1}{2}} = C^{\frac{1}{2}}\mathbb{E}(1-1/C)$, where $\mathbb{E}$ is the complete elliptic integral of the first kind ($\frac{1}{2}\pi < \mathbb{E} < 1$) and $C = \exp Q$ ($1 < C < \infty$). For $\mu = 2$ the condition for the existence of solutions is $\delta^{\frac{1}{2}} = C^{\frac{1}{2}}[2\mathbb{E}(m)-\mathbb{K}(m)]$, where $m = (C-1)/2C$ and $\mathbb{K}$ is the complete elliptic integral of the second kind. Numerical solutions for $\mu = 0.5$ are shown on figure 4(*a*). It has been checked that on the range $0.2 \leqslant \mu \leqslant 2$ the results are not very sensitive to the value of μ.

Equation (2.34*c*), which is representative of the intermediate-Prandtl-number case, has been solved numerically for a value of $\mu = 0.4$, corresponding to $Pr = 1$. The amplitude profiles are shown on figure 4(*b*). In order to compare the results of the integration of (2.34*a, c*) we have drawn on figure 5 the maximum amplitude as a function of δ for the value $\mu = 0.4$.

In the limit $Pr \to 0$ the numerical integration of (2.34*b*) gives quite different results. The condition for solutions to be possible demands that $\delta^{\frac{1}{2}}$ takes its values between $\mu^{\frac{1}{2}}$ and $\frac{1}{2}\pi$. Since $\mu \geqslant 6$ we get solutions in the range $\frac{1}{2}\pi < \delta^{\frac{1}{2}} < \mu^{\frac{1}{2}}$.

We should mention that (2.36) is now replaced by

$$\log(1-A^2)+\frac{1}{\mu}\log(1-\mu B^2) = Q,$$

where $Q < 0$. Let $C = \exp Q$, with $0 < C < 1$. The amplitude profiles for different values of C and $\mu = 10$ are shown on figure 6. When $C \to 0$ the profile tends to a constant slope.

Equations (2.34*a*–*c*) can be put in the general form

$$A_{\xi\xi}+A+\epsilon(A^2A_{\xi\xi}+\mu AA_\xi^2) = A_\tau, \tag{2.37}$$

where $\epsilon = \pm 1$ and μ is either positive or negative. Equation (2.37) describes the coupling between modes of same parity, for instance

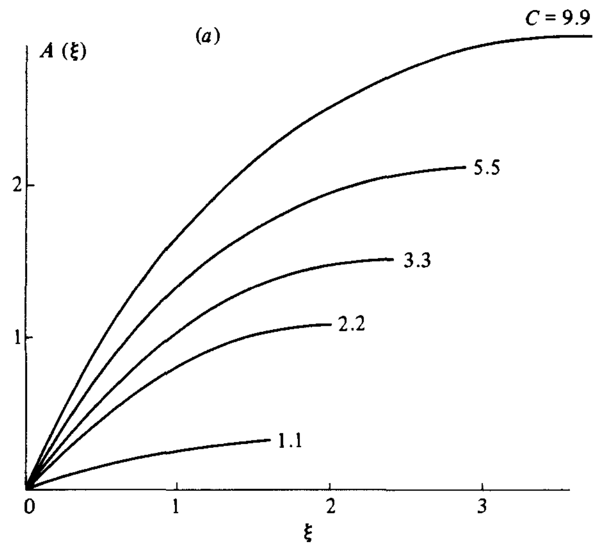
$$A = \sum_n A_n \cos\tfrac{1}{2}n\pi\xi \quad \text{or} \quad A = \sum_n B_n \sin n\pi\xi.$$

The numerical results shown on figures 4(*a, b*) can be viewed as the superposition of the two lowest even modes:

$$A = A_1\cos\tfrac{1}{2}\pi\xi + A_2\cos\tfrac{3}{2}\pi\xi,$$

A_1 and A_2 being of opposite sign. More than two modes are necessary to fit the results of figure 6.

We have shown in this section that owing to the symmetry of the horizontal modes, the nonlinear terms in the equation for the vertical amplitude $A(Z)$ are either quadratic or cubic, and the corresponding solutions are respectively asymmetrical or symmetrical about $Z = 0$. The same distinction between symmetrical and asym-



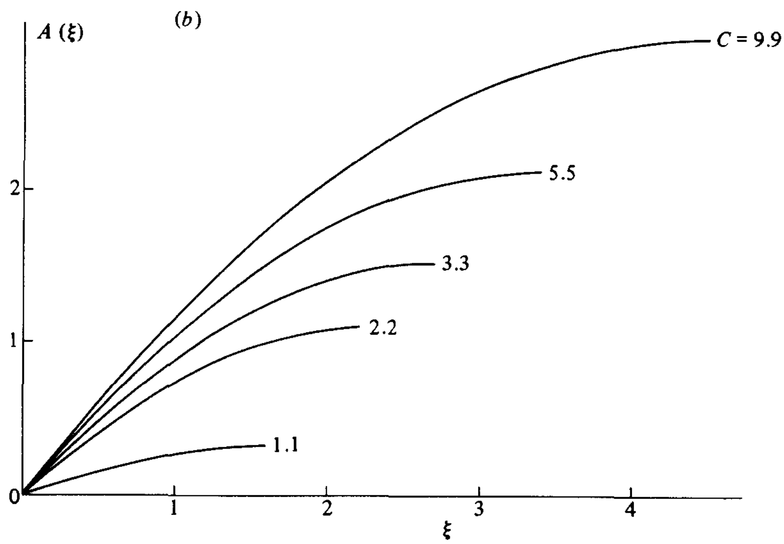


FIGURE 4. Numerical solutions of $A_{\xi\xi}+A+A^2A_{\xi\xi}+\mu AA_\xi^2 = 0$ with $A = 0$ at $\xi = \pm\delta^{\frac{1}{2}}$: (a) $\mu = 0.5$; (b) $\mu = -0.4$.

metrical solutions also arises in the field of nonlinear convection between poorly conducting boundary, where Chapman & Proctor (1980) have derived an equation of the form

$$A_{\xi\xi}+A-A^3+\alpha AA_\xi = 0, \tag{2.38}$$

which, except for the cubic term, is identical with (2.17) for $\alpha = 1$. When $\alpha = 0$ the solutions of (2.38) are symmetrical and (2.38) can be solved in term of elliptic functions, whereas for $\alpha \neq 0$ the solutions are asymmetrical.

We shall see in §3 how to transpose the results of this section to the case of a vertical circular cylinder.

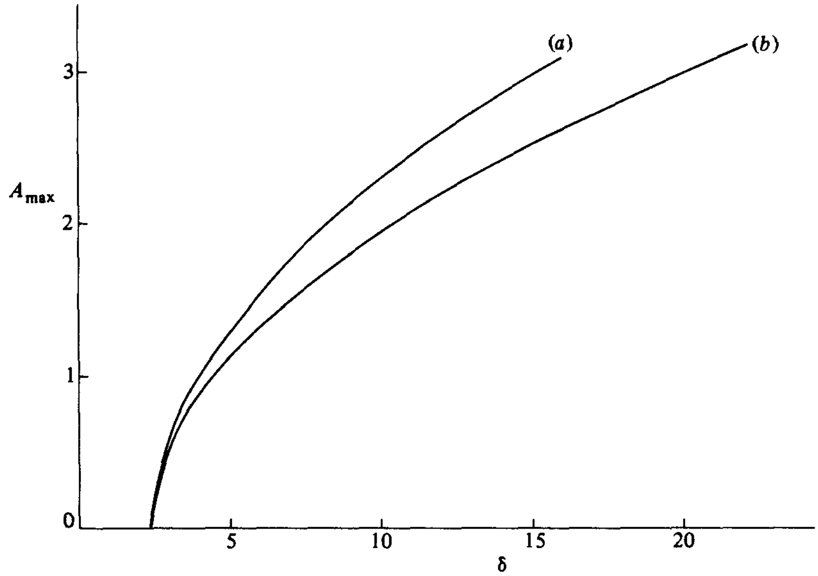


FIGURE 5. The maximum amplitude as a function of δ: (a) $\mu = 0.5$; (b) $\mu = -0.4$.

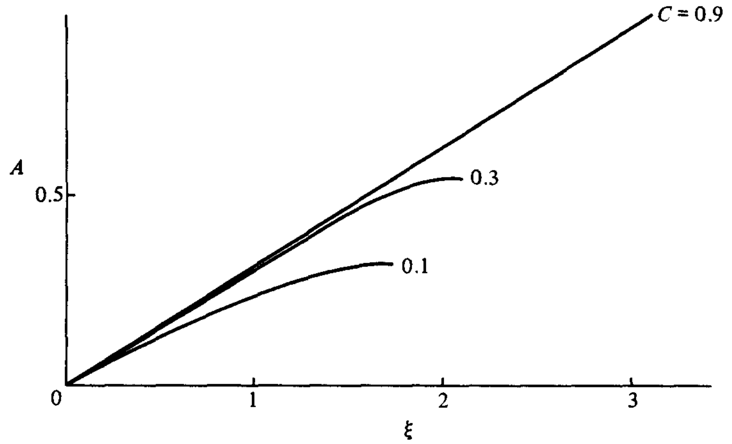


FIGURE 6. Numerical solutions of $A_{\xi\xi}+A-A^2A_{\xi\xi}-\mu AA_\xi^2 = 0$ with $A = 0$ at $\xi = \pm\delta^{\frac{1}{2}}$. Solutions are possible for $\frac{1}{2}\pi < \delta^{\frac{1}{2}} < \mu^{\frac{1}{2}}$.

3. The vertical circular cylinder

In a cylindrical geometry it is convenient to deal with the following representation of $\boldsymbol{v}$:

$$\boldsymbol{v} = \boldsymbol{\nabla}\times\psi\boldsymbol{e}+\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\phi\boldsymbol{e}, \tag{3.1}$$

and to replace the set of equations (2.1) by

$$(Pr^{-1}\partial_t-\Delta)\Delta_2\psi = Pr^{-1}N_v^{(a)}, \tag{3.2a}$$

$$(Pr^{-1}\partial_t-\Delta)\Delta\Delta_2\phi+\Delta_2\theta = -Pr^{-1}N_v^{(b)}, \tag{3.2b}$$

$$(\partial_t-\Delta)\theta = -Ra\,\Delta_2\phi-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\theta, \tag{3.2c}$$

with

$$\Delta = \Delta_2+\partial_z^2, \quad \Delta_2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2},$$

and where $N_v^{(a)}$ and $N_v^{(b)}$ stand for the nonlinear terms coming from $(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}$:

$$N_v^{(a)} = \boldsymbol{e}\cdot(\nabla\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}), \quad N_v^{(b)} = \boldsymbol{e}\cdot(\nabla\times\nabla\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}).$$

The change of variable $z = \lambda Z$ is made, together with the decomposition of all the physical quantities into two contributions:

$$f = f^{(0)} + \tilde{f}, \tag{3.3}$$

where the quadrivector notation $f \equiv (\psi, \phi, \theta, Ra)$ has been used. In (3.3) $f^{(0)}$ stands for the Ostroumov solutions and it is assumed that the perturbations can be expanded in inverse powers of λ. For an infinite cylinder $(\lambda \to \infty)$ the time-independent equations reduce to

$$\Delta_2 \psi^{(0)} = 0, \tag{3.4a}$$

$$[\Delta_2^2 - Ra^{(0)}]\, v_z^{(0)} = 0, \tag{3.4b}$$

with

$$\theta^{(0)} = -\Delta_2 v_z^{(0)}. \tag{3.4c}$$

The rigid boundary conditions associated with (3.4*a*) at $r = 1$ are $\psi^{(0)} = \partial_r \psi^{(0)} = 0$, so that $\psi^{(0)}$ is identically null. The solutions for the vertical component of the velocity and the temperature divide in two classes following the thermal boundary condition and the angular mode.

(i) Conducting lateral boundary and $n \neq 0$:

$$v_z^{(0)} = A\, \mathrm{e}^{\mathrm{i}n\varphi} J_n(kr), \quad \theta^{(0)} = Ak^2\, \mathrm{e}^{\mathrm{i}n\varphi} J_n(kr), \tag{3.5a, b}$$

with

$$J_n(k) = 0. \tag{3.5c}$$

(ii) Insulating lateral boundary or $n = 0$:

$$v_z^{(0)} = A\, \mathrm{e}^{\mathrm{i}n\varphi}\left[\frac{J_n(kr)}{J_n(k)} - \frac{I_n(kr)}{I_n(k)}\right], \quad \theta^{(0)} = Ak^2\, \mathrm{e}^{\mathrm{i}n\varphi}\left[\frac{J_n(kr)}{J_n(k)} + \frac{I_n(kr)}{I_n(k)}\right], \tag{3.6a, b}$$

with

$$I_n(k)\, J_n'(k) + J_n(k)\, I_n'(k) = 0, \tag{3.6c}$$

where J_n and I_n are respectively the nth Bessel and modified Bessel functions of the first kind. The values of k satisfying (3.5*c*) and (3.6*c*) are given in Gershuni & Zukhovitskii (1972). To simplify the notations it will be useful to introduce two functions of the radial variable:

$$F_{\pm}^{(n)}(kr) = \left\{\begin{array}{ll} \dfrac{J_n(kr)}{J_n(k)} \pm \dfrac{I_n(kr)}{I_n(k)} & \text{(insulating sidewalls)}, \\[2ex] J_n(kr) & \text{(conducting sidewalls)}. \end{array}\right\} \tag{3.7}$$

In the above notation the integer n characterizes the angular mode of flow; for example, $n = 0$ corresponds to the axisymmetrical mode and $n = 1$ to the first diametrically antisymmetrical mode. We begin by investigating the axisymmetrical mode, for which the components of the convective velocity are given by

$$v_r = \frac{\partial^2 \phi}{\partial z\, \partial r}, \quad v_z = -\Delta_0 \phi, \quad v_\varphi = 0,$$

with

$$\Delta_0 = r^{-1}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r}.$$

It must be noticed that for $n = 0$ the solutions of (3.4) are the same for either type of thermal boundary condition, and are given by (3.6*a*, *b*). When the expansions (3.3) are substituted into (3.2), one gets for the perturbed velocity

$$[\Delta_0^2 - Ra^{(0)}]\,\tilde{v}_z = -\partial_t \theta^{(0)} - N_\theta + 3D_Z^2\theta^{(0)}\lambda^{-2} + \widetilde{Ra}\,v_z^{(0)}, \tag{3.8a}$$

with

$$N_\theta = \lambda^{-1}k^2 A A_Z[|F_+^{(0)\prime}|^2 + F_+^{(0)}F_-^{(0)}], \tag{3.8b}$$

where the prime denotes derivative with respect to r and the limit $Pr \rightarrow \infty$ has been taken. The solvability condition for (3.8*a*) takes a form analogous to (2.12):

$$3\lambda^{-2}A_{ZZ}I_1 + \widetilde{Ra}\,I_2 A + \lambda^{-1}I_3 A A_Z = I_1 A_t, \tag{3.9}$$

with

$$I_1 = k^2\int_0^1 |F_+^{(0)}|^2\, r\,\mathrm{d}r = \frac{1+2J_1(k)}{kJ_0(k)}, \tag{3.10a}$$

$$I_2 = \int_0^1 F_+^{(0)}F_-^{(0)}\, r\,\mathrm{d}r = \left[\frac{J_1(k)}{J_0(k)}\right]^2, \tag{3.10b}$$

$$I_3 = -k^2\int_0^1 F_+^{(0)}[|F_+^{(0)\prime}|^2 + F_+^{(0)}F_-^{(0)}]\, r\,\mathrm{d}r. \tag{3.10c}$$

After integrating by parts, one can show that I_3 takes a simple form

$$I_3 = \frac{3}{2}\int_0^1 |F_+^{(0)}|^2\, F_-^{(0)}\, r\,\mathrm{d}r,$$

which has been calculated numerically for the lowest radial mode, $k = 4.611$, and is

$$I_3 = -0.6029.$$

All the parameters can then be eliminated in (3.9), leading to the equation (2.17) derived in §2 for the case of even modes between vertical parallel plates.

The situation is quite different for the case of the diametrically antisymmetrical modes, $n \geqslant 1$, for which one allows A to be a complex function:

$$A = |A|\,\mathrm{e}^{\mathrm{i}\alpha},$$

where both $|A|$ and the phase α are slowly varying functions of the vertical coordinate. Then the physical quantities, for instance the convective velocity, are now of the form

$$v_z = (A\,\mathrm{e}^{\mathrm{i}n\varphi} + A^*\,\mathrm{e}^{-\mathrm{i}n\varphi})\,w(r). \tag{3.11}$$

As a consequence, the nonlinearities in the equation for the perturbed quantities $\tilde{\phi}$, $\tilde{\psi}$ and $\tilde{\theta}$ are either independent of φ or are of the form $\mathrm{e}^{\pm 2\mathrm{i}n\varphi}$, so that the orthogonality relations with the adjoint solutions of (3.9) are automatically satisfied, with the result that $I_3 = 0$. The method used to obtain the nonlinear equation satisfied by A closely follows the one described in §2 for the case of odd modes between vertical parallel plates. The main difficulty specific to the circular geometry comes from the fact that there are no multiplication rules for the Bessel functions as there are for trigonometric or hyperbolic functions. When the perturbed quantities are expanded in powers of λ^{-1}, for example

$$\tilde{\phi} = \phi^{(1)}\lambda^{-1} + \phi^{(2)}\lambda^{-2} + \dots$$

and analogous expansions for $\tilde{\psi}$, $\tilde{\theta}$ and $\widetilde{Ra}$, The quantities $\phi^{(i)}, \dots$, with $i \geqslant 1$, cannot be determined analytically, and approximate methods must be used. Our starting point will be Ostroumov approximation, to which we superimpose a vertical modulation together with allowing a phase shift along the vertical axis of the cylinder, so that the fundamental quantities take the form

$$\theta^{(0)} = k^2(A\,\mathrm{e}^{\mathrm{i}n\varphi}+A^*\,\mathrm{e}^{-\mathrm{i}n\varphi})\,F_+^{(n)}(kr), \tag{3.12a}$$

$$v_z^{(0)} = (A\,\mathrm{e}^{\mathrm{i}n\varphi}+A^*\,\mathrm{e}^{-\mathrm{i}n\varphi})\,F_-^{(n)}(kr). \tag{3.12b}$$

Contrary to what happens when $n = 0$, the vertical vorticity $\Delta_2\psi$, which is null in the Ostroumov approximation, does not remain null at higher orders in the λ^{-1} power expansion. The first-order solutions satisfy

$$\Delta_2^2\,\psi^{(1)} = 0, \tag{3.13}$$

and the boundary conditions associated with the rigid case, $v_r = v_\varphi = 0$ at $r = 1$, provide the boundary condition for $\psi^{(1)}$:

$$\left.\begin{aligned} \frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\varphi} &= -\frac{\partial^2\phi^{(0)}}{\partial z\,\partial r}, \\ \frac{\partial\psi^{(1)}}{\partial r} &= \frac{1}{r}\frac{\partial^2\phi^{(0)}}{\partial z\,\partial\varphi} \end{aligned}\right\} \quad \text{at} \quad r = 1. \tag{3.14}$$

This leads us to seek solutions of (3.13) of the form

$$\psi^{(1)} = \mathrm{i}(A_Z\,\mathrm{e}^{\mathrm{i}n\varphi}-A_Z^*\,\mathrm{e}^{-\mathrm{i}n\varphi})\,(C_1 r^n + C_2 r^{n+2}), \tag{3.15}$$

where the constant real coefficients C_1 and C_2 take the following values:

(i) insulating case

$$C_1 = -C_2 = -\frac{n}{2k^2}F_+^{(n)}(k); \tag{3.16a}$$

(ii) conducting case

$$C_1 = \frac{n+2}{2nk}F_+^{(n)\prime}(k), \quad C_2 = -\frac{1}{2k}F_+^{(n)\prime}(k). \tag{3.16b}$$

We now have all the elements necessary to calculate the nonlinear terms to first order in λ^{-1}:

$$N^{(1)} = \left(\frac{1}{r}\frac{\partial\psi^{(1)}}{\partial\varphi}+\frac{\partial\phi^{(0)}}{\partial z\,\partial r}\right)\frac{\partial\theta^{(0)}}{\partial r}+\frac{1}{r}\left(\frac{1}{r}\frac{\partial^2\phi^{(0)}}{\partial z\,\partial\varphi}-\frac{\partial\psi^{(1)}}{\partial r}\right)\frac{\partial\theta^{(0)}}{\partial\varphi}+v_z^{(0)}\frac{\partial\theta^{(0)}}{\partial z},$$

which is of the form

$$N^{(1)} = (AA_Z\mathrm{e}^{2\mathrm{i}n\varphi}+A_Z^*A^*\,\mathrm{e}^{-2\mathrm{i}n\varphi})\,N^{(1,\,2)}+(A_Z A^*+A_Z^* A)\,N^{(1,\,0)}. \tag{3.17}$$

This suggests that we seek the solutions of

$$[\Delta_2^2 - Ra^{(0)}]\,v_z^{(1)} = -N^{(1)} \tag{3.18}$$

in the form

$$v_z^{(1)} = (AA_Z\,\mathrm{e}^{2\mathrm{i}n\varphi}+A_Z^*A^*\,\mathrm{e}^{-2\mathrm{i}n\varphi})\,w^{(1,\,2)}+(A_Z A^*+A_Z^* A)\,w^{(1,\,0)}, \tag{3.19}$$

where the radial functions $w^{(1,\,2)}$ and $w^{(1,\,0)}$ can be determined by the Galerkin method, the trial functions being the eigenfunctions of the linear problem, so that

$$w^{(1,\,2)} = \sum_{i=1}^{N} D_i^{(2)}F_-^{(2n)}(k_i r), \tag{3.19a}$$

$$w^{(1,\,0)} = \sum_{i=1}^{N} D_i^{(0)}F_-^{(0)}(k_i r), \tag{3.19b}$$

with

$$D_i^{(2)} = [k_i^4 - Ra^{(0)}]^{-1}\left[\int_0^1 N^{(1,2)}F_+^{(2n)}(k_i r)\, r\,\mathrm{d}r\right]\left[\int_0^1 F_+^{(2n)}F_-^{(2n)} r\,\mathrm{d}r\right]^{-1}$$

and

$$D_i^{(0)} = [k_i^4 - Ra^{(0)}]^{-1}\left[\int_0^1 N^{(1,0)}F_+^{(0)}(k_i r)\, r\,\mathrm{d}r\right]\left[\int_0^1 F_+^{(0)}F_-^{(0)} r\,\mathrm{d}r\right]^{-1},$$

where the functions $F_\pm$ are defined in (3.7). In a similar way we introduce also $\theta^{(1,2)}$ and $\theta^{(1,0)}$, since $\theta^{(1)}$ admits the same decomposition as $v_z^{(1)}$. We also deduce that

$$\psi^{(2)} = \mathrm{i}[(AA_Z)_Z\,\mathrm{e}^{2\mathrm{i}n\varphi} - (A_Z^* A^*)_Z\,\mathrm{e}^{-2\mathrm{i}n\varphi}]\,\psi^{(2,2)}(r). \tag{3.20}$$

The contribution $\psi^{(2,0)}$ independent of φ is of the form

$$\psi^{(2,0)} = B_0 r^2 + B_1,$$

which only contributes to $v_\varphi^{(2,0)} \equiv \partial\psi^{(2,0)}/\partial r = 2B_0 r$, and owing to the rigid boundary condition on the sidewalls it turns out that $B_0 = 0$, so that $\psi^{(2,0)}$ disappears from the calculation. Then the nonlinear terms at order λ^{-2} can be calculated, and the solvability condition for the equation satisfied by $v_z^{(2)}$ takes the form

$$A_{\xi\xi} + A + \mu_0 A|A|^2_{\xi\xi} + \mu_1 A_\xi|A|^2_\xi + \mu_2 A|A_\xi|^2 + \mu_3 A^*(AA_\xi)_\xi = A_\tau, \tag{3.21}$$

where $\mu_0 = \pm 1$ and the explicit form of the μ_i coefficients $(i > 1)$ is given in Appendix B. The boundary conditions are

$$A = 0 \quad \text{at} \quad \xi = \pm\delta^{\frac{1}{2}}.$$

When there is a constant phase along the vertical axis, so that A is real, (3.21) reduces to (2.37). Since no variational formulation exists for (3.21), solutions like $A(\varphi - \omega t)$ with a phase rotating at the frequency ω cannot be excluded. Let us examine the case for which the complex equation (3.21) separates into two real equations by writing

$$A = A(\xi)\,\mathrm{e}^{\mathrm{i}[\alpha(\xi)+\omega\tau]}.$$

We obtain

$$\ddot{W} - \dot{\alpha}^2 W + W + (2\mu_0 + 2\mu_1 + \mu_2 + \mu_3)\, W\dot{W}^2 + (2\mu_0 + \mu_3)\, W^2\ddot{W} + \dot{\alpha}^2 W^3(\mu_2 - 2\mu_3) = 0 \tag{3.22a}$$

and

$$\ddot{\alpha} W + 2\dot{\alpha}\dot{W} + 2\mu_1\dot{\alpha} W^2\dot{W} + \mu_3 W(4\dot{\alpha} W\dot{W} + \ddot{\alpha} W^2) = \omega W, \tag{3.22b}$$

where a dot means differentiation with respect to ξ. Multiplying (3.22b) by W and integrating by parts over ξ from $-\delta^{\frac{1}{2}}$ to $+\delta^{\frac{1}{2}}$ gives

$$\omega = 2\mu_1\left[\int_{-\delta^{\frac{1}{2}}}^{\delta^{\frac{1}{2}}} \dot{\alpha} W^3\dot{W}\,\mathrm{d}z\right]\left[\int_{-\delta^{\frac{1}{2}}}^{\delta^{\frac{1}{2}}} W^2\,\mathrm{d}z\right]^{-1}.$$

Consequently $\omega = 0$ if either $\mu_1 = 0$ or if the phase is constant along the vertical axis ($\dot{\alpha} = 0$). The spatial variation of the phase induces modification of the convective flow pattern. This point is illustrated on figure 7, where we have represented a diametrically antisymmetrical mode with a constant phase (a) and with an arbitrary variation of the phase. There is experimental evidence for helical convective patterns bearing some analogy with figure 7 and associated with an oscillatory behaviour in solidification processes (Azouni 1977). It has been reported to me that in the field of photochemical reactions, convective patterns in the form of a figure of eight have been observed in high-evaporation-rate cooling cells (figure 3d), giving evidence of non-constant-phase solutions. Simplifications occur in the resolution of (3.22a, b) when degenerate cases

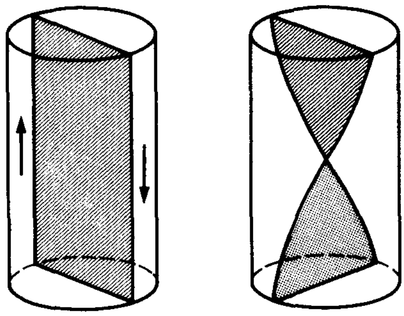


FIGURE 7. Schematic representation of a diametrically antisymmetrical mode in a cylinder. (a) With constant phase along the axis: the vertical nodal plane which divides the cylinder in zones of respectively downward and upward flow is represented. (b) Example of an arbitrary helical variation of the phase: the nodal plane is now twisted.

are considered. Let, for instance, $\mu_1 = \mu_2 = 0$. Then multiplying (3.22*b*) by W and integrating gives

$$\dot{\alpha}(W^2+\mu_3 W^4) = H. \tag{3.23}$$

Using (3.23), then multiplying (3.22*a*) by $\dot{W}$ and integrating, we obtain

$$\dot{W}^2+W^2+(2\mu_0+\mu_3)\,W^2\dot{W}^2+\frac{H^2}{W^2+\mu_3 W^4} = E,$$

where H and E are the constants of integration. Therefore W is given by inversion of the expression

$$Z = \int_{W_0}^{W}\left[\frac{(1+\mu_3 W^2)\,[1+(2\mu_0+\mu_3)\,W^2]}{(E-W^2)\,(W^2+\mu_3 W^4)-H^2}\right]^{\frac{1}{2}} W\,\mathrm{d}W. \tag{3.24}$$

It can be shown that (3.24) reduces to an elliptic integral of the second kind when $\mu_3 = 0$. Unfortunately, when all these simplifications are made we must take $H = 0$ in (3.23) to prevent divergence of the phase at the boundaries, and the corresponding solutions are those with a constant phase. The possibility of steady solutions with a non-constant phase along the axis of the cylinder cannot be ruled out, however, and is presently under investigation.

4. Conclusion

The Ostroumov solution for convection in infinitely long vertical channels does not allow the generation of nonlinear terms in a Landau-type expansion. When the endwalls at the top and the bottom are taken into account, so that the nonlinear terms cannot be discarded in the equations, the solutions of the convective problem are in general obtained by numerical methods. Focusing on non-axisymmetrical modes in vertical cylinders of moderate aspect ratio, Rosenblat (1982) derived nonlinear evolution equations of the Landau type. His method, which consists of expanding the field quantities in series of eigenfunctions of the linear stability problem, has also been used by Lyubimov, Putin & Chernatynskii (1979) in the case of a Hele-Shaw cell. The present contribution differs from these previous works in that the spatial variation as well as the time evolution of the amplitude are taken into account. In vertical channels the vertical amplitude is held constant when the assumption of an infinite height is made, while it is considered as a slowly varying function of the vertical coordinate as well as time when the height is finite. The disparity between horizontal and vertical length-scales has been exploited to develop an expansion scheme in powers of the small radius-to-height ratio. Our calculations give the result that the nonlinear terms in the differential equation satisfied by the amplitude A are either cubic or quadratic in A, depending on the symmetry of the horizontal spatial mode of flow. Moreover, these nonlinear terms contain derivatives with respect to Z, contrary to what happens for the case of horizontal structures. These equations have been solved numerically in the steady state, showing that the vertical structure of the convective flow in the weakly nonlinear regime is asymmetrical about $Z = 0$ when quadratic terms are present, and symmetrical when the nonlinear terms are cubic. Few experimental data in high vertical cells with vertical temperature gradient are available for comparison with our results. We should mention the work of Olson & Rosenberger (1979) in vertical cylinders with $h = 6R$, where there is evidence for an asymmetrical mode of flow (figure 3*c*) due to a mode–mode coupling and similar to what we get in solving (2.17).

The existence of two kinds of nonlinearities in the amplitude equation owing to the symmetry of the horizontal modes bears some analogy with the existence of two convective patterns, rolls or hexagons, in horizontal fluid layers. In the latter case Palm (1960) has shown that the breaking of the vertical symmetry is responsible for the formation of hexagonal cells.

Although the Prandtl-number dependence only appears in the case of a plane vertical layer, it seems likely that the essential features of the problem are contained in (2.17) and (3.21). Recent contributions (Siggia & Zippelius 1981) have emphasized the non-trivial role of a finite Prandtl number when horizontal layers of fluid are considered. In that case a finite Prandtl number is necessary to allow the generation of vertical vorticity. The situation is different in a cylindrical geometry, since vertical vorticity is always present even in the limit $Pr \rightarrow \infty$ (equation (3.15)). The finite-Prandtl-number effect is a change of the numerical values of the μ_i coefficients in (3.21). As suggested by experimental findings (Olson & Rosenberger 1979), it is expected that a finite Prandtl number will change the threshold for the onset of oscillatory instability, but this problem is well beyond the scope of the present analysis.

In conclusion, the existence of amplitude equations for vertical channels gives some hope of answering the following questions for the specific case of circular cylinders.

(i) Are there stationary solutions with no constant phase? The difficulty lies in the treatment of the boundary layers.

(ii) Are the solutions with a constant phase stable for disturbances having a rotating phase?

The author acknowledges with gratitude many discussions with Y. Pomeau during the course of this work.

Appendix A. Derivation of the boundary conditions for A

We follow closely the derivation given by Brown & Stewartson (1977) for the case of a horizontal layer of fluid. In the neighbourhood of a horizontal boundary the adjustment to the mechanical and thermal conditions is taking place rapidly, so that the assumption of a small $A^{-1}\,\partial A/\partial z$ is no longer valid. Then the nonlinear terms in (2.1) can be neglected near the threshold of the instability, and the equation to be solved in the boundary layer is

$$[(\partial_x^2+\partial_z^2)^3 - Ra\,\partial_x^2]\,v_z = 0. \tag{A 1}$$

A further simplification is made by taking $Ra = Ra^{(0)}$, the relative error in so doing is $O(\lambda^{-2})$. For conducting sidewalls the odd modes are given by

$$v_z = w(z)\sin \pi x, \tag{A 2}$$

where $w(z)$ satisfies

$$(\partial_z^4 - 3\pi^2\partial_z^2 + 3\pi^4)\,\partial_z^2 w = 0, \tag{A 3}$$

whose general solution is

$$w(z) = C_0 + C_1 z + \sum_{i=2}^{5} C_i\,e^{\mu_i \pi z}, \tag{A 4}$$

and the μ_i are solutions of the algebraic equation

$$\mu^4 - 3\mu^2 + 3 = 0. \tag{A 5}$$

Near the wall $z = -\lambda$ we need consider only those of the μ_i that contribute to an exponentially decaying solution for $z \to \infty$. A complete solution for the boundary layer is

$$v_z = (C_0 + C_1 z + C_2\,e^{\mu\pi z} + C_3\,e^{\mu^*\pi z})\sin \pi x, \tag{A 6a}$$

$$v_x = (C_1 + \mu\pi C_2\,e^{\mu\pi z} + \mu^*\pi C_3\,e^{\mu^*\pi z})\,\pi^{-1}\cos \pi x, \tag{A 6b}$$

$$\theta = [C_0 + C_1 z + C_2(\mu^2-1)^2\,e^{\mu\pi z} + C_3(\mu^{*2}-1)^2\,e^{\mu^*\pi z}]\,\pi^2 \sin \pi x. \tag{A 6c}$$

The boundary conditions $v_z = v_x = \theta = 0$ at $z = -\lambda$ leads to the following algebraic system for the coefficients:

$$\bar{C}_0 = C_0 - C_1\lambda, \quad C_1, \quad \bar{C}_2 = C_2\,e^{\mu\pi\lambda}, \quad \bar{C}_3 = C_3\,e^{\mu^*\pi\lambda}.$$

After dropping the overbars we get

$$\left.\begin{aligned} C_0 + C_2 + C_3 &= 0,\\ C_1 + \mu\pi C_2 + \mu^*\pi C_3 &= 0,\\ C_2\mu^2(\mu^2-2) + C_3\mu^{*2}(\mu^{*2}-2) &= 0.\end{aligned}\right\} \tag{A 7}$$

We have three equations connecting four constants C_i, and the remaining condition must come from matching (A 6) as $z \to \infty$, with the expansion of A near $Z = -\lambda$

$$A(z) = A(-\lambda) + \lambda^{-1} z A'(-\lambda) + \dots, \tag{A 8}$$

where A' denotes $\lambda\partial_g A$ and is $O(A)$. After eliminating C_2 and C_3 in (A 7), one gets a linear relation between C_0 and C_1

$$C_1 = \pi C_0[Re\,\mu + \sqrt{3}\,\mathrm{Im}\,\mu]. \tag{A 9}$$

The matching of (A 8) with (A 6a) shows that C_1 is of relative order λ^{-1}, and so we may set $C_0 = 0$ in (A 9) as a first approximation. Hence the appropriate boundary condition on A is $A = 0$.

In the case of a vertical cylinder the equations to be solved are

$$\left.\begin{aligned} (\Delta^3 - Ra\,\Delta_2)\,\Delta_2\phi &= 0,\\ \Delta\Delta_2\psi &= 0,\end{aligned}\right\} \tag{A 10}$$

with

$$\theta = \Delta^2\phi.$$

Near a horizontal wall we seek the solutions of (A 10) in the form

$$\phi = J_n(kr) f(z) \cos n\varphi,$$
$$\psi = (a_1 r^n + a_2 r^{n+2})\, g(z) \sin n\varphi.$$

The boundary conditions
$$v_r = v_\varphi = 0 \quad \text{at} \quad r = 1$$
will be easily satisfied by assuming $g(z) = \mathrm{d}f/\mathrm{d}z$. Then $f(z)$ satisfies an equation analogous to (A 3):

$$(\mathrm{d}_z^4 - 3k^2\,\mathrm{d}_z^2 + 3k^4)\,\mathrm{d}_z^2 f = 0,$$

with the conditions

$$f = \mathrm{d}_z f = (\mathrm{d}_z^2 - k^2)^2 f = 0 \quad \text{at} \quad z = 0.$$

From now the calculations are exactly the same as those for a plane vertical layer and will not be repeated here.

Appendix B. Calculation of the μ_i coefficients which appear in front of the nonlinear terms in the complex amplitude equation for the non-axisymmetrical modes in cylinders

Our starting point is the solvability condition at order λ^{-2}:

$$3I_1 A_{ZZ} + Ra^{(2)} I_2 A + I_3 A[(A_Z A^*)_Z + (A_Z^* A)_Z] + I_4 A_Z (A^* A_Z + A_Z^* A) \\ + I_5 A A_Z A_Z^* + I_6 A^* (A A_Z)_Z = 0, \quad \text{(B 1)}$$

where the I_m $(m = 1, \dots, 6)$ are integrals over the r-variable with I_1 and I_2 already defined in (3.10a, b). For $m \geqslant 3$ they are given by the following expressions:

$$I_3 = \int_0^1 [w^{(0)}\theta^{(1,0)} + v_r^{(2,0)}\,\partial_r \theta^{(0)}]\,\theta^{(0)} r\,\mathrm{d}r, \quad \text{(B 2}a)$$

$$I_4 = \int_0^1 [w^{(1,0)}\theta^{(0)} + v_r^{(1)}\,\partial_r \theta^{(1,0)}]\,\theta^{(0)} r\,\mathrm{d}r, \quad \text{(B 2}b)$$

$$I_5 = \int_0^1 \left[w^{(1,2)}\theta^{(0)} + v_r^{(1)}\,\partial_r \theta^{(1,2)} + 2\theta^{(1,2)}\frac{v_\varphi^{(1)}}{r}\right]\theta^{(0)} r\,\mathrm{d}r, \quad \text{(B 2}c)$$

$$I_6 = \int_0^1 \left[w^{(0)}\theta^{(1,2)} + v_r^{(2,2)}\,\partial_r \theta^{(0)} + \theta^{(0)}\frac{v_\varphi^{(2,2)}}{r}\right]\theta^{(0)} r\,\mathrm{d}r. \quad \text{(B 2}d)$$

We dimensionalize as follows:

$$Z = \left[\frac{3I_1}{Ra^{(2)} I_2}\right]^{\frac{1}{2}} \xi, \quad A = \left[\frac{3I_1}{|I_3|}\right]^{\frac{1}{2}} \bar{A}.$$

After dropping the overbar, (B 1) takes the form (3.21), where

$$\mu_0 = \frac{I_3}{|I_3|}, \quad \mu_1 = \frac{I_4}{|I_3|}, \quad \mu_2 = \frac{I_5}{|I_3|}, \quad \mu_3 = \frac{I_6}{|I_3|}. \quad \text{(B 3)}$$

Expressions (B 2a–d) are valid when the Prandtl number is infinite. When this is not the case, the I_m depend upon the Prandtl number.

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